

# Asymptotic behavior of homogeneous additive functionals of the solutions of Itô stochastic differential equations with nonregular dependence on parameter

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**Abstract** We study the asymptotic behavior of mixed functionals of the form  $I_T(t) = F_T(\xi_T(t)) + \int_0^t g_T(\xi_T(s)) d\xi_T(s)$ ,  $t \geq 0$ , as  $T \rightarrow \infty$ . Here  $\xi_T(t)$  is a strong solution of the stochastic differential equation  $d\xi_T(t) = a_T(\xi_T(t)) dt + dW_T(t)$ ,  $T > 0$  is a parameter,  $a_T = a_T(x)$  are measurable functions such that  $|a_T(x)| \leq C_T$  for all  $x \in \mathbb{R}$ ,  $W_T(t)$  are standard Wiener processes,  $F_T = F_T(x)$ ,  $x \in \mathbb{R}$ , are continuous functions,  $g_T = g_T(x)$ ,  $x \in \mathbb{R}$ , are locally bounded functions, and everything is real-valued. The explicit form of the limiting processes for  $I_T(t)$  is established under very nonregular dependence of  $g_T$  and  $a_T$  on the parameter  $T$ .

**Keywords** Diffusion-type processes, asymptotic behavior of additive functionals, nonregular dependence on the parameter

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## 1 Introduction

Consider the Itô stochastic differential equation

$$d\xi_T(t) = a_T(\xi_T(t)) dt + dW_T(t), \quad t \geq 0, \quad \xi_T(0) = x_0, \quad (1)$$

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where  $T > 0$  is a parameter,  $a_T(x)$ ,  $x \in \mathbb{R}$ , are real-valued measurable functions such that for some constants  $L_T > 0$  and for all  $x \in \mathbb{R}$   $|a_T(x)| \leq L_T$ , and  $W_T = \{W_T(t), t \geq 0\}$ ,  $T > 0$ , is a family of standard Wiener processes defined on a complete probability space  $(\Omega, \mathfrak{F}, P)$ .

It is known from Theorem 4 in [19] that, for any  $T > 0$  and  $x_0 \in \mathbb{R}$ , equation (1) possesses a unique strong pathwise solution  $\xi_T = \{\xi_T(t), t \geq 0\}$ , and this solution is a homogeneous strong Markov process.

We suppose that the drift coefficient  $a_T(x)$  in equation (1) can have a very non-regular dependence on the parameter. For example, the drift coefficient can be of “ $\delta$ ”-type sequence at some points  $x_k$  as  $T \rightarrow \infty$ , or it can be equal to  $\sqrt{T} \sin((x - x_k)\sqrt{T})$ , or it can have degeneracies of some other types. Such a nonregular dependence of the coefficients in equation (1) first appeared in [5] and [4], where the limit behavior of the normalized unstable solution of Itô stochastic differential equation as  $t \rightarrow \infty$  was investigated. In those papers, a special dependence of the coefficients  $a_T(x) = \sqrt{T}a(x\sqrt{T})$  on the parameter  $T$  was considered in the case where  $a(x)$  is an absolutely integrable function on  $\mathbb{R}$ . Assume that this is the case and let  $\int_{\mathbb{R}} a(x) dx = \lambda$ . The sufficiency of the condition  $\lambda = 0$  for the asymptotic equivalence of distributions  $\xi_T$  and  $W_T(t)$  is established in [5], and the necessity of this condition is proved in [1]. If  $\lambda \neq 0$ , then we can deduce from [4] that the distributions of the solution  $\xi_T$  of equation (1) weakly converge as  $T \rightarrow \infty$  to the corresponding distributions of the Markov process  $\hat{\xi}(t) = l(\zeta(t))$ , where  $l(x) = c_1 x$  for  $x > 0$  and  $l(x) = c_2 x$  for  $x \leq 0$ ;  $\zeta(t)$  is a strong solution of the Itô equation  $d\zeta(t) = \bar{\sigma}(\zeta(t)) dW(t)$ , where  $\bar{\sigma}(x) = \sigma_1$  for  $x > 0$  and  $\bar{\sigma}(x) = \sigma_2$  for  $x < 0$ , and  $\int_0^t P\{|\zeta(s)| = 0\} ds = 0$ . The explicit form of the transition density of the process  $\hat{\xi}(t)$  is obtained. Moreover, in [10], it is proved that

$$\hat{\xi}(t) = x_0 + \beta(t) + W(t),$$

where  $\beta(t)$  is a certain functional of  $\zeta(t)$ , and the necessity of the condition  $\lambda \neq 0$  is established for the weak convergence as  $T \rightarrow \infty$  of the solution  $\xi_T$  of equation (1) to the process  $\hat{\xi}(t)$ .

Furthermore, in [5] and [4], a probabilistic method to study the “awkward” term  $\sqrt{T} \int_0^t a(\xi_T(s)\sqrt{T}) ds$  in equation (1) is developed. This method uses a representation of this “awkward” term through a family of continuous functions  $\Phi_T(x)$  of  $\xi_T(t)$  and a family of martingales  $\int_0^t \Phi_T'(\xi_T(s)) dW_T(s)$ , with the further application of the Itô formula. After the mentioned transformations, according to this method, we can apply Skorokhod’s convergent subsequence principle for  $\xi_{T_n}(t)$  and  $W_{T_n}(t)$  (see [17], Chapter I, §6) in order to pass to the limit in the resulting representation.

Note that this method is also used in the present paper to study the asymptotic behavior of integral functionals.

It is known from [2], §16, that the asymptotic behavior of the solution  $\xi_T$  of equation (1) is closely related to the asymptotic behavior of harmonic functions, that is, functions satisfying the following ordinary differential equation almost everywhere (a.e.) with respect to the Lebesgue measure:

$$f_T'(x)a_T(x) + \frac{1}{2}f_T''(x) = 0.$$

It is obvious that the functions  $f_T(x)$  have the form

$$f_T(x) = c_T^{(1)} \int_0^x \exp \left\{ -2 \int_0^u a_T(v) dv \right\} du + c_T^{(2)}, \quad (2)$$

where  $c_T^{(1)}$  and  $c_T^{(2)}$  are some families of constants.

The latter functions possess the continuous derivatives  $f'_T(x)$ , and their second derivatives  $f''_T(x)$  exist almost everywhere with respect to the Lebesgue measure and are locally integrable. Note that  $c_T^{(1)}$  are normalizing constants and  $c_T^{(2)}$  are centralizing constants in the limit theorems (see [18], §6). Further, for simplicity, we assume that in (2),  $c_T^{(1)} \equiv 1$  and  $c_T^{(2)} \equiv 0$ .

In this paper, we assume for the coefficient  $a_T(x)$  of equation (1) that there exists a family of functions  $G_T(x)$ ,  $x \in \mathbb{R}$ , with continuous derivatives  $G'_T(x)$  and locally integrable second derivatives  $G''_T(x)$  a.e. with respect to the Lebesgue measure such that, for all  $T > 0$  and  $x \in \mathbb{R}$ , the following inequalities hold:

$$(A_1) \quad \left( G'_T(x) a_T(x) + \frac{1}{2} G''_T(x) \right)^2 + (G'_T(x))^2 \leq C(1 + (G_T(x))^2),$$

$$|G_T(x_0)| \leq C.$$

Suppose additionally that the functions  $G_T(x)$ ,  $x \in \mathbb{R}$ , introduced by condition (A<sub>1</sub>) satisfy the following assumptions:

- (i) There exist constants  $C > 0$  and  $\alpha > 0$  such that  $|G_T(x)| \geq C|x|^\alpha$ .
- (ii) There exist a bounded function  $\psi(|x|)$  and a constant  $m \geq 0$  such that  $\psi(|x|) \rightarrow 0$  as  $|x| \rightarrow 0$  and, for all  $x \in \mathbb{R}$  and  $T > 0$  and for any measurable bounded set  $B$ , the following inequality holds:

$$(A_2) \quad \int_0^x f'_T(u) \left( \int_0^u \frac{\chi_B(G_T(v))}{f'_T(v)} dv \right) du \leq \psi(\lambda(B)) [1 + |x|^m],$$

where  $\chi_B(v)$  is the indicator function of a set  $B$ ,  $\lambda(B)$  is the Lebesgue measure of  $B$ , and  $f'_T(x)$  is the derivative of the function  $f_T(x)$  defined by equality (2).

Let  $\{G_T\}$  be the class of the functions  $G_T(x)$ ,  $x \in \mathbb{R}$ , satisfying conditions (A<sub>1</sub>) and (i)–(ii). The class of equations of the form (1) whose coefficients  $a_T(x)$  admit  $G_T(x)$ ,  $x \in \mathbb{R}$ , from the class  $\{G_T\}$  will be denoted by  $K(G_T)$ . It is easy to understand that class  $K(G_T)$  does not depend on the constants  $c_T^{(1)}$  and  $c_T^{(2)}$  in representation (2).

It is clear that if there exist constants  $\delta > 0$  and  $C > 0$  such that  $0 < \delta \leq f'_T(x) \leq C$  for all  $x \in \mathbb{R}$ ,  $T > 0$ , then the corresponding equations (1) belong to the class  $K(G_T)$  for  $G_T(x) = f_T(x)$ . We denote this subclass as  $K_1$ . Note that the class  $K(G_T)$  contains in particular the equations for which, at some points  $x_k$ , we have the convergence  $f'_T(x_k) \rightarrow \infty$  or the convergence  $f'_T(x_k) \rightarrow 0$  as  $T \rightarrow \infty$ . For example, consider equation (1) with  $a_T(x) = \frac{c_0 T x}{1+x^2 T}$ . It is easy to obtain that  $f'_T(x) = \frac{1}{(1+x^2 T)^{c_0}}$ , and if  $c_0 > -\frac{1}{2}$ , then such equations belong to the class  $K(G_T)$  with  $G_T(x) = x^2$  (here, at points  $x \neq 0$ , we have  $f'_T(x) \rightarrow 0$  for  $c_0 > 0$ ,  $f'_T(x) \rightarrow \infty$  for  $-\frac{1}{2} < c_0 < 0$ , and  $f'_T(x) \equiv 1$  for  $c_0 = 0$ ).

For the class of equations  $K(G_T)$ , we study the asymptotic behavior as  $T \rightarrow \infty$  of the distributions of the following functionals:

$$\begin{aligned}\beta_T^{(1)}(t) &= \int_0^t g_T(\xi_T(s)) ds, & \beta_T^{(2)}(t) &= \int_0^t g_T(\xi_T(s)) dW_T(s), \\ I_T(t) &= F_T(\xi_T(t)) + \int_0^t g_T(\xi_T(s)) dW_T(s), & \beta_T(t) &= \int_0^t g_T(\xi_T(s)) d\xi_T(s),\end{aligned}$$

where the processes  $\xi_T(t)$ ,  $W_T(t)$  are related via equation (1),  $g_T(x)$  is a family of measurable locally bounded real-valued functions, and  $F_T(x)$  is a family of continuous real-valued functions.

This paper is a continuation of [13–15]. Note that the behavior of the distributions of functionals  $\beta_T^{(1)}(t)$ ,  $\beta_T^{(2)}(t)$  for the solutions  $\xi_T$  of equations (1) from the class  $K_1$  is studied in [6] and [9]. The case where  $W_T(t)$  is replaced with  $\eta_T(t)$ , where  $\eta_T(t)$  is a family of continuous martingales with the characteristics  $\langle \eta_T \rangle(t) \rightarrow t$  as  $T \rightarrow \infty$ , was studied in [8]. Paper [7] was devoted to a discrete analogue of the results from [8]. A similar problem for the functionals  $I_T(t)$  in the case of equation (1) with  $a_T(x) \equiv 0$  was considered in [18] and in [11], for the class  $K_1$ . In [13–15], the behavior of the distributions of the functionals  $\beta_T^{(1)}(t)$ ,  $\beta_T^{(2)}(t)$ , and  $I_T(t)$  with a special dependence of the drift coefficients  $a_T(x) = \sqrt{T}a(x\sqrt{T})$  on the parameter  $T$  is considered, mainly in the case where  $|xa(x)| \leq C$  for all  $x \in \mathbb{R}$ . The behavior of the distributions of functionals  $\beta_T^{(1)}(t)$  was studied in [13],  $\beta_T^{(2)}(t)$  was studied in [14], and  $I_T(t)$  was investigated in [15]. A more detailed review of the known results in this area is presented in [13–15]. Note that the functionals  $\beta_T^{(1)}(t)$ ,  $\beta_T^{(2)}(t)$ , and  $\beta_T(t)$  are particular cases of the functional  $I_T(t)$  (see [15], Lemma 4.1).

**Remark 1.1.** In this paper, we often apply the Itô formula to the process  $\Phi(\xi_T(t))$ , where  $\xi_T(t)$  is a solution of equation (1), the derivative  $\Phi'(x)$  of the function  $\Phi(x)$  is assumed to be continuous, and the second derivative  $\Phi''(x)$  is assumed to exist a.e. with respect to the Lebesgue measure and to be locally integrable. Then it follows from [3] that with probability one, for all  $t \geq 0$ , the following equality holds:

$$\begin{aligned}\Phi(\xi_T(t)) - \Phi(x_0) &= \int_0^t \left( \Phi'(\xi_T(s))a_T(\xi_T(s)) + \frac{1}{2} \Phi''(\xi_T(s)) \right) ds \\ &\quad + \int_0^t \Phi'(\xi_T(s)) dW_T(s).\end{aligned}$$

**Remark 1.2.** Let  $\xi_T$  be a solution of equation (1), and  $G_T(x)$  be a family of functions satisfying condition  $(A_1)$ . Theorem 1 from [12] implies that the family of the processes  $\{\zeta_T(t) = G_T(\xi_T(t)), t \geq 0\}$  is weakly compact. The proof of this result is based on the equality

$$\zeta_T(t) = G_T(x_0) + \int_0^t \left( G_T'(\xi_T(s))a_T(\xi_T(s)) + \frac{1}{2} G_T''(\xi_T(s)) \right) ds + \eta_T(t), \quad (3)$$

where

$$\eta_T(t) = \int_0^t G_T'(\xi_T(s)) dW_T(s), \quad \zeta_T(t) = G_T(\xi_T(t)).$$

In turn, the latter equality follows from Remark 1.1. In addition, it is established in the proof of Theorem 1 from [12] that for any constants  $L > 0$  and  $\varepsilon > 0$ ,

$$\begin{aligned} \lim_{N \rightarrow \infty} \overline{\lim_{T \rightarrow \infty}} \sup_{0 \leq t \leq L} \mathbb{P}\{|\lambda_T(t)| > N\} &= 0, \\ \lim_{h \rightarrow 0} \overline{\lim_{T \rightarrow \infty}} \sup_{|t_1 - t_2| \leq h; t_i \leq L} \mathbb{P}\{|\lambda_T(t_2) - \lambda_T(t_1)| > \varepsilon\} &= 0, \end{aligned} \quad (4)$$

and that, for any  $k > 1$  and for certain constants  $C_k$  and  $C$ ,

$$\mathbb{E} \sup_{0 \leq t \leq L} |\lambda_T(t)|^k \leq C_k, \quad \mathbb{E} |\lambda_T(t_2) - \lambda_T(t_1)|^4 \leq C|t_2 - t_1|^2, \quad (5)$$

where  $\lambda = \eta$  or  $\lambda = \zeta$  (see [2], §6, Theorem 4).

**Remark 1.3.** Here and throughout the paper, the weak convergence of the processes means the weak convergence in the uniform topology of the space of continuous functions  $C[0, L]$  for any  $L > 0$ . The processes that have continuous trajectories with probability 1 will be simply called continuous.

The paper is organized as follows. Section 2 contains the statements of the main results. In Section 3, they are proved. Auxiliary results are collected in Section 4.

## 2 Statement of the main results

In what follows, we denote by  $C$ ,  $L$ ,  $N$ ,  $C_N$  any constants that do not depend on  $T$  and  $x$ . Assume that, for certain locally bounded functions  $q_T(x)$  and any constant  $N > 0$ , the following condition holds:

$$(A_3) \quad \lim_{T \rightarrow \infty} \sup_{|x| \leq N} f'_T(x) \left| \int_0^x \frac{q_T(v)}{f'_T(v)} dv \right| = 0,$$

where  $f'_T(x)$  is the derivative of the function  $f_T(x)$  defined by Eq. (2).

**Theorem 2.1.** *Let  $\xi_T$  be a solution of Eq. (1) from the class  $K(G_T)$  and  $G_T(x_0) \rightarrow y_0$  as  $T \rightarrow \infty$ . Assume that there exist measurable locally bounded functions  $a_0(x)$  and  $\sigma_0(x)$  such that:*

1. *the functions*

$$q_T^{(1)}(x) = G'_T(x)a_T(x) + \frac{1}{2}G''_T(x) - a_0(G_T(x))$$

*and*

$$q_T^{(2)}(x) = (G'_T(x))^2 - \sigma_0^2(G_T(x))$$

*satisfy assumption (A<sub>3</sub>);*

2. *the Itô equation*

$$\zeta(t) = y_0 + \int_0^t a_0(\zeta(s)) ds + \int_0^t \sigma_0(\zeta(s)) d\hat{W}(s) \quad (6)$$

*has a unique weak solution.*

*Then the stochastic process  $\zeta_T(t) = G_T(\xi_T(t))$  weakly converges, as  $T \rightarrow \infty$ , to the solution  $\zeta(t)$  of Eq. (6).*

**Theorem 2.2.** Let  $\xi_T$  be a solution of Eq. (1) from the class  $K(G_T)$ , and let the assumptions of Theorem 2.1 hold. Assume that, for measurable locally bounded functions  $g_T(x)$ , there exists a measurable locally bounded function  $g_0(x)$  such that the function

$$q_T(x) = g_T(x) - g_0(G_T(x))$$

satisfies assumption  $(A_3)$ . Then the stochastic process  $\beta_T^{(1)}(t) = \int_0^t g_T(\xi_T(s)) ds$  weakly converges, as  $T \rightarrow \infty$ , to the process

$$\beta^{(1)}(t) = \int_0^t g_0(\zeta(s)) ds,$$

where  $\zeta(t)$  is a solution of Eq. (6).

**Theorem 2.3.** Let  $\xi_T$  be a solution of Eq. (1) from the class  $K(G_T)$ , and let the assumptions of Theorem 2.1 hold. Assume that, for measurable locally bounded functions  $g_T(x)$ , there exists a measurable locally bounded function  $g_0(x)$  such that

$$(A_4) \quad \lim_{T \rightarrow \infty} \sup_{|x| \leq N} \left| f'_T(x) \int_0^x \frac{g_T(v)}{f'_T(v)} dv - g_0(G_T(x)) G'_T(x) \right| = 0$$

for all  $N > 0$ . Then the stochastic process  $\beta_T^{(1)}(t) = \int_0^t g_T(\xi_T(s)) ds$  weakly converges, as  $T \rightarrow \infty$ , to the process

$$\tilde{\beta}^{(1)}(t) = 2 \left( \int_{y_0}^{\zeta(t)} g_0(x) dx - \int_0^t g_0(\zeta(s)) \sigma_0(\zeta(s)) d\hat{W}(s) \right),$$

where  $\zeta(t)$  and the Wiener process  $\hat{W}(t)$  are related via Eq. (6).

**Theorem 2.4.** Let  $\xi_T$  be a solution of equation (1) from the class  $K(G_T)$ , and let the assumptions of Theorem 2.1 hold. Assume that, for measurable locally bounded functions  $g_T(x)$ , there exists a measurable locally bounded function  $g_0(x)$  such that the function

$$q_T(x) = (g_T(x) - g_0(G_T(x)) G'_T(x))^2$$

satisfies assumption  $(A_3)$ . Then the stochastic process  $\beta_T^{(2)}(t) = \int_0^t g_T(\xi_T(s)) dW_T(s)$ , where  $\xi_T(t)$  and  $W_T(t)$  are related via Eq. (1), weakly converges, as  $T \rightarrow \infty$ , to the process

$$\beta^{(2)}(t) = \int_0^t g_0(\zeta(s)) d\zeta(s) - \int_0^t g_0(\zeta(s)) a_0(\zeta(s)) ds,$$

where  $\zeta(t)$  is a solution of Eq. (6).

**Theorem 2.5.** Let  $\xi_T$  be a solution of Eq. (1) from the class  $K(G_T)$ , and let the assumptions of Theorem 2.1 hold. Assume that, for continuous functions  $F_T(x)$  and locally bounded measurable functions  $g_T(x)$ , there exist a continuous function  $F_0(x)$  and locally bounded measurable function  $g_0(x)$  such that, for all  $N > 0$ ,

$$\lim_{T \rightarrow \infty} \sup_{|x| \leq N} |F_T(x) - F_0(G_T(x))| = 0,$$

and let the functions  $g_T(x)$  and  $g_0(x)$  satisfy the assumptions of Theorem 2.4. Then the stochastic process

$$I_T(t) = F_T(\xi_T(t)) + \int_0^t g_T(\xi_T(s)) dW_T(s),$$

where  $\xi_T(t)$  and  $W_T(t)$  are related via Eq. (1), weakly converges, as  $T \rightarrow \infty$ , to the process

$$I_0(t) = F_0(\zeta(t)) + \int_0^t g_0(\zeta(s)) \sigma_0(\zeta(s)) d\hat{W}(s),$$

where  $\zeta(t)$  and the Wiener process  $\hat{W}(t)$  are related via Eq. (6).

The next theorem principally follows from [11]; however, we provide its proof for the reader's convenience and completeness of the results.

**Theorem 2.6.** *Let  $\xi_T$  be a solution of Eq. (1) from the class  $K(G_T)$  for  $G_T(x) = f_T(x)$ , and let  $0 < \delta \leq f'_T(x) \leq C$  and  $f_T(x_0) \rightarrow y_0$  as  $T \rightarrow \infty$ . Also, let  $\zeta_T(t) = f_T(\xi_T(t))$ ,*

$$I_T(t) = F_T(\xi_T(t)) + \int_0^t g_T(\xi_T(s)) dW_T(s),$$

*$F_T(x)$  be continuous functions,  $g_T(x)$  be locally square-integrable functions, and the processes  $\xi_T(t)$  and  $W_T(t)$  be related via Eq. (1).*

*The two-dimensional process  $(\zeta_T(t), I_T(t))$  weakly converges, as  $T \rightarrow \infty$ , to the process  $(\zeta(t), I(t))$ , where  $I(t) = F_0(\zeta(t)) + \int_0^t g_0(\zeta(s)) d\zeta(s)$ , and  $\zeta(t)$  is a weak solution of the Itô equation  $\zeta(t) = y_0 + \int_0^t \sigma_0(\zeta(s)) dW(s)$ , if and only if there exist constants  $c_T^{(1)}$  and  $c_T^{(2)}$  in (2) such that, as  $T \rightarrow \infty$ :*

1. *for all  $x$ ,*

$$\int_0^{\varphi_T(x)} \frac{[f'_T(v)]^2 - \sigma_0^2(f_T(v))}{f'_T(v)} dv \rightarrow 0,$$

*where  $\varphi_T(x)$  is the inverse function of the function  $f_T(x)$ ;*

2. *for all  $N > 0$ ,*

$$\sup_{|x| \leq N} |F_T(x) + f_T(x) - F_0(f_T(x))| \rightarrow 0$$

*and*

$$\int_{-N}^N \frac{|g_T(x) - f'_T(x)[1 + g_0(f_T(x))]|^2}{f'_T(x)} dx \rightarrow 0.$$

### 3 Proof of the main results

Proof of Theorem 2.1. Rewrite Eq. (3) as

$$\zeta_T(t) = G_T(x_0) + \int_0^t a_0(\zeta_T(s)) ds + \alpha_T^{(1)}(t) + \eta_T(t), \quad (7)$$

where

$$\alpha_T^{(1)}(t) = \int_0^t q_T^{(1)}(\xi_T(s)) ds, \quad q_T^{(1)}(x) = G'_T(x)a_T(x) + \frac{1}{2} G''_T(x) - a_0(G_T(x)).$$

The functions  $q_T^{(1)}(x)$  satisfy the conditions of Lemma 4.2. Thus, for any  $L > 0$ ,

$$\sup_{0 \leq t \leq L} |\alpha_T^{(1)}(t)| \xrightarrow{P} 0 \quad (8)$$

as  $T \rightarrow \infty$ . It is clear that  $\eta_T(t)$  is a family of continuous martingales with quadratic characteristics

$$\langle \eta_T \rangle(t) = \int_0^t (G'_T(\xi_T(s)))^2 ds = \int_0^t \sigma_0^2(\zeta_T(s)) ds + \alpha_T^{(2)}(t), \quad (9)$$

where

$$\alpha_T^{(2)}(t) = \int_0^t q_T^{(2)}(\xi_T(s)) ds, \quad q_T^{(2)}(x) = (G'_T(x))^2 - \sigma_0^2(G_T(x)).$$

The functions  $q_T^{(2)}(x)$  satisfy the conditions of Lemma 4.2. Thus, for any  $L > 0$ ,

$$\sup_{0 \leq t \leq L} |\alpha_T^{(2)}(t)| \xrightarrow{P} 0 \quad (10)$$

as  $T \rightarrow \infty$ .

We have that relations (4) and (5) hold for the processes  $\zeta_T(t)$  and  $\eta_T(t)$ , and, according to (8) and (10), these relations hold for the processes  $\alpha_T^{(k)}(t)$ ,  $k = 1, 2$ , as well. This means that we can apply Skorokhod's convergent subsequence principle (see [17], Chapter I, §6) for the process  $(\zeta_T(t), \eta_T(t), \alpha_T^{(1)}(t), \alpha_T^{(2)}(t))$ : given an arbitrary sequence  $T'_n \rightarrow \infty$ , we can choose a subsequence  $T_n \rightarrow \infty$ , a probability space  $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{P})$ , and a stochastic process  $(\tilde{\zeta}_{T_n}(t), \tilde{\eta}_{T_n}(t), \tilde{\alpha}_{T_n}^{(1)}(t), \tilde{\alpha}_{T_n}^{(2)}(t))$  defined on this space such that its finite-dimensional distributions coincide with those of the process  $(\zeta_{T_n}(t), \eta_{T_n}(t), \alpha_{T_n}^{(1)}(t), \alpha_{T_n}^{(2)}(t))$  and, moreover,

$$\tilde{\zeta}_{T_n}(t) \xrightarrow{\tilde{P}} \tilde{\zeta}(t), \quad \tilde{\eta}_{T_n}(t) \xrightarrow{\tilde{P}} \tilde{\eta}(t), \quad \tilde{\alpha}_{T_n}^{(1)}(t) \xrightarrow{\tilde{P}} \tilde{\alpha}^{(1)}(t), \quad \tilde{\alpha}_{T_n}^{(2)}(t) \xrightarrow{\tilde{P}} \tilde{\alpha}^{(2)}(t)$$

for all  $0 \leq t \leq L$ , where  $\tilde{\zeta}(t)$ ,  $\tilde{\eta}(t)$ ,  $\tilde{\alpha}^{(1)}(t)$ ,  $\tilde{\alpha}^{(2)}(t)$  are some stochastic processes.

Evidently, relations (8) and (10) imply that  $\tilde{\alpha}^{(k)}(t) \equiv 0$ ,  $k = 1, 2$ , a.s. According to (5), the processes  $\tilde{\zeta}(t)$  and  $\tilde{\eta}(t)$  are continuous. Moreover, applying Lemma 4.5 together with Eqs. (7) and (9), we obtain that

$$\begin{aligned} \tilde{\zeta}_{T_n}(t) &= G_{T_n}(x_0) + \int_0^t a_0(\tilde{\zeta}_{T_n}(s)) ds + \tilde{\alpha}_{T_n}^{(1)}(t) + \tilde{\eta}_{T_n}(t), \\ \langle \tilde{\eta}_{T_n} \rangle(t) &= \int_0^t \sigma_0^2(\tilde{\zeta}_{T_n}(s)) ds + \tilde{\alpha}_{T_n}^{(2)}(t), \end{aligned} \quad (11)$$

where  $\tilde{\zeta}_{T_n}(t) \xrightarrow{\tilde{P}} \tilde{\zeta}(t)$ ,  $\tilde{\eta}_{T_n}(t) \xrightarrow{\tilde{P}} \tilde{\eta}(t)$ ,  $\sup_{0 \leq t \leq L} |\tilde{\alpha}_{T_n}^{(k)}(t)| \xrightarrow{\tilde{P}} 0$ ,  $k = 1, 2$ , as  $T_n \rightarrow \infty$ . In addition, it is established in [12] that, for any constants  $L > 0$  and  $\varepsilon > 0$ ,

$$\lim_{h \rightarrow 0} \overline{\lim_{T_n \rightarrow \infty}} \tilde{P} \left\{ \sup_{|t_1 - t_2| \leq h; t_i \leq L} |\tilde{\zeta}_{T_n}(t_2) - \tilde{\zeta}_{T_n}(t_1)| > \varepsilon \right\} = 0.$$

Using the latter convergence and (11), we conclude that, for any constants  $L > 0$  and  $\varepsilon > 0$ ,



$$\lim_{h \rightarrow 0} \overline{\lim}_{T_n \rightarrow \infty} \tilde{\mathbb{P}} \left\{ \sup_{|t_1 - t_2| \leq h; t_i \leq L} |\tilde{\eta}_{T_n}(t_2) - \tilde{\eta}_{T_n}(t_1)| > \varepsilon \right\} = 0.$$

Therefore, according to the well-known result of Prokhorov [16], we conclude that

$$\sup_{0 \leq t \leq L} |\tilde{\zeta}_{T_n}(t) - \tilde{\zeta}(t)| \xrightarrow{\tilde{\mathbb{P}}} 0 \quad \text{and} \quad \sup_{0 \leq t \leq L} |\tilde{\eta}_{T_n}(t) - \tilde{\eta}(t)| \xrightarrow{\tilde{\mathbb{P}}} 0$$

as  $T_n \rightarrow \infty$ . According to Lemma 4.3, we can pass to the limit in (11) and obtain

$$\tilde{\zeta}(t) = y_0 + \int_0^t a_0(\tilde{\zeta}(s)) ds + \tilde{\eta}(t), \quad (12)$$

where  $\tilde{\eta}(t)$  is a continuous martingale with the quadratic characteristic

$$\langle \tilde{\eta} \rangle(t) = \int_0^t \sigma_0^2(\tilde{\zeta}(s)) ds.$$

Now, it is well known that the latter representation provides the existence of a Wiener process  $\hat{W}(t)$  such that

$$\tilde{\eta}(t) = \int_0^t \sigma_0(\tilde{\zeta}(s)) d\hat{W}(s). \quad (13)$$

Thus, the process  $(\tilde{\zeta}(t), \hat{W}(t))$  satisfies Eq. (6), and the process  $\tilde{\zeta}_{T_n}(t)$  weakly converges, as  $T_n \rightarrow \infty$ , to the process  $\tilde{\zeta}(t)$ . Since the subsequence  $T_n \rightarrow \infty$  is arbitrary and since a solution of Eq. (6) is weakly unique, the proof of the Theorem 2.1 is complete.

**Proof of Theorem 2.2.** It is clear that, for all  $t > 0$ , with probability one,

$$\beta_T^{(1)}(t) = \int_0^t g_0(\zeta_T(s)) ds + \alpha_T(t),$$

where  $\alpha_T(t) = \int_0^t q_T(\xi_T(s)) ds$  and  $q_T(x) = g_T(x) - g_0(G_T(x))$ .

The functions  $q_T(x)$  satisfy the conditions of Lemma 4.2. Thus, for any  $L > 0$ ,

$$\sup_{0 \leq t \leq L} |\alpha_T(t)| \xrightarrow{\mathbb{P}} 0$$

as  $T \rightarrow \infty$ . Similarly to (11), we obtain the equality

$$\tilde{\beta}_{T_n}^{(1)}(t) = \int_0^t g_0(\tilde{\zeta}_{T_n}(s)) ds + \tilde{\alpha}_{T_n}(t), \quad (14)$$

where  $\tilde{\zeta}_{T_n}(t) \xrightarrow{\tilde{\mathbb{P}}} \tilde{\zeta}(t)$  and  $\sup_{0 \leq t \leq L} |\tilde{\alpha}_{T_n}(t)| \xrightarrow{\tilde{\mathbb{P}}} 0$  as  $T_n \rightarrow \infty$ . The process  $\tilde{\zeta}(t)$  is a solution of Eq. (12), whereas by Lemma 4.5 the finite-dimensional distributions of the stochastic process  $\beta_{T_n}^{(1)}(t)$  coincide with those of the process  $\tilde{\beta}_{T_n}^{(1)}(t)$ .

Using Lemma 4.3 and Eq. (14), we conclude that

$$\sup_{0 \leq t \leq L} \left| \tilde{\beta}_{T_n}^{(1)}(t) - \int_0^t g_0(\tilde{\zeta}(s)) ds \right| \xrightarrow{\tilde{\mathbb{P}}} 0$$

as  $T_n \rightarrow \infty$ . Thus, the process  $\beta_{T_n}^{(1)}(t)$  weakly converges, as  $T_n \rightarrow \infty$ , to the process  $\beta^{(1)}(t) = \int_0^t g_0(\zeta(s)) ds$ , where  $\zeta(t)$  is a solution of Eq. (6). Since the subsequence

$T_n \rightarrow \infty$  is arbitrary and since a solution  $\zeta(t)$  of Eq. (6) is weakly unique, the proof of Theorem 2.2 is complete.

Proof of Theorem 2.3. Consider the function

$$\Phi_T(x) = 2 \int_0^x f'_T(u) \left( \int_0^u \frac{g_T(v)}{f'_T(v)} dv \right) du.$$

Applying the Itô formula to the process  $\Phi_T(\xi_T(t))$ , where  $\xi_T(t)$  is a solution of Eq. (1), we get that

$$\begin{aligned} \beta_T^{(1)}(t) &= \Phi_T(\xi_T(t)) - \Phi_T(x_0) - \int_0^t \Phi'_T(\xi_T(s)) dW_T(s) \\ &= 2 \int_{x_0}^{\xi_T(t)} g_0(G_T(u)) G'_T(u) du - 2 \int_0^t g_0(\zeta_T(s)) G'_T(\xi_T(s)) dW_T(s) \\ &\quad + 2 \int_{x_0}^{\xi_T(t)} \hat{q}_T(u) du - 2 \int_0^t \hat{q}_T(\xi_T(s)) dW_T(s) \\ &= 2 \int_{G_T(x_0)}^{\zeta_T(t)} g_0(u) du - 2 \int_0^t g_0(\zeta_T(s)) d\eta_T(s) + \gamma_T^{(1)}(t) - \gamma_T^{(2)}(t), \end{aligned}$$

where

$$\begin{aligned} \gamma_T^{(1)}(t) &= 2 \int_{x_0}^{\xi_T(t)} \hat{q}_T(u) du, \quad \gamma_T^{(2)}(t) = 2 \int_0^t \hat{q}_T(\xi_T(s)) dW_T(s), \\ \hat{q}_T(x) &= \left( f'_T(x) \int_0^x \frac{g_T(v)}{f'_T(v)} dv - g_0(G_T(x)) G'_T(x) \right). \end{aligned}$$

Denote  $P_{NT} = \mathbf{P}\{\sup_{0 \leq t \leq L} |\xi_T(t)| > N\}$ . It is clear that, for any constants  $\varepsilon > 0$ ,  $N > 0$ , and  $L > 0$ , we have the inequalities

$$\begin{aligned} \mathbf{P} \left\{ \sup_{0 \leq t \leq L} |\gamma_T^{(1)}(t)| > \varepsilon \right\} &\leq P_{NT} + \frac{2}{\varepsilon} \mathbf{E} \sup_{0 \leq t \leq L} \left| \int_{x_0}^{\xi_T(t)} \hat{q}_T(u) du \right| \chi_{\{|\xi_T(t)| \leq N\}} \\ &\leq P_{NT} + \frac{2}{\varepsilon} \int_{-N}^N |\hat{q}_T(u)| du \leq P_{NT} + \frac{4}{\varepsilon} N \sup_{|x| \leq N} |\hat{q}_T(x)| \end{aligned}$$

and

$$\begin{aligned} \mathbf{P} \left\{ \sup_{0 \leq t \leq L} |\gamma_T^{(2)}(t)| > \varepsilon \right\} &\leq P_{NT} + \frac{4}{\varepsilon^2} \mathbf{E} \sup_{0 \leq t \leq L} \left| \int_0^t \hat{q}_T(\xi_T(s)) \chi_{\{|\xi_T(s)| \leq N\}} dW_T(s) \right|^2 \\ &\leq P_{NT} + \frac{16}{\varepsilon^2} \mathbf{E} \int_0^L \hat{q}_T^2(\xi_T(s)) \chi_{\{|\xi_T(s)| \leq N\}} ds \leq P_{NT} + \frac{16}{\varepsilon^2} L \sup_{|x| \leq N} |\hat{q}_T^2(x)|. \end{aligned}$$

The inequality  $|G_T(x)| \geq C|x|^\alpha$ ,  $\alpha > 0$ , together with convergence (4), implies that

$$\lim_{N \rightarrow \infty} \overline{\lim}_{T \rightarrow \infty} P_{NT} = 0. \quad (15)$$

Thus, using the conditions of Theorem 2.3, we get the convergence

$$\sup_{0 \leq t \leq L} |\gamma_T^{(k)}(t)| \xrightarrow{P} 0, \quad k = 1, 2,$$

as  $T \rightarrow \infty$ .

The same arguments as we used establishing (11) yield that

$$\tilde{\beta}_{T_n}^{(1)}(t) = 2 \int_{G_{T_n}(x_0)}^{\tilde{\zeta}_{T_n}(t)} g_0(u) du - 2 \int_0^t g_0(\tilde{\zeta}_{T_n}(s)) d\tilde{\eta}_{T_n}(s) + \tilde{\gamma}_{T_n}^{(1)}(t) - \tilde{\gamma}_{T_n}^{(2)}(t), \quad (16)$$

where

$$\begin{aligned} \sup_{0 \leq t \leq L} |\tilde{\zeta}_{T_n}(t) - \tilde{\zeta}(t)| &\xrightarrow{\tilde{P}} 0, & \sup_{0 \leq t \leq L} |\tilde{\eta}_{T_n}(t) - \tilde{\eta}(t)| &\xrightarrow{\tilde{P}} 0, \\ \sup_{0 \leq t \leq L} |\tilde{\gamma}_{T_n}^{(k)}(t)| &\xrightarrow{\tilde{P}} 0, & k = 1, 2, \end{aligned}$$

as  $T_n \rightarrow \infty$  for all  $L > 0$ . According to (12) with  $\tilde{\eta}(t)$  defined in (13), the process  $(\tilde{\zeta}(t), \tilde{W}(t))$  satisfies Eq. (6).

By Lemma 4.5 the finite-dimensional distributions of the stochastic process  $\beta_{T_n}^{(1)}(t)$  coincide with those of the process  $\tilde{\beta}_{T_n}^{(1)}(t)$ . Using Lemma 4.3, we can pass to the limit as  $T_n \rightarrow \infty$  in (16) and obtain

$$\sup_{0 \leq t \leq L} |\tilde{\beta}_{T_n}^{(1)}(t) - \tilde{\beta}^{(1)}(t)| \xrightarrow{\tilde{P}} 0 \quad (17)$$

as  $T_n \rightarrow \infty$ , where

$$\begin{aligned} \tilde{\beta}^{(1)}(t) &= 2 \int_{y_0}^{\tilde{\zeta}(t)} g_0(u) du - 2 \int_0^t g_0(\tilde{\zeta}(s)) d\tilde{\eta}(s) \\ &= 2 \int_{y_0}^{\tilde{\zeta}(t)} g_0(u) du - 2 \int_0^t g_0(\tilde{\zeta}(s)) d\tilde{\zeta}(s) + 2 \int_0^t g_0(\tilde{\zeta}(s)) a_0(\tilde{\zeta}(s)) ds, \end{aligned}$$

and  $\tilde{\zeta}(t)$  is a solution of Eq. (6). Therefore, we have that Theorem 2.3 holds for the process  $\beta_{T_n}^{(1)}(t)$  as  $T_n \rightarrow \infty$ . Since the subsequence  $T_n \rightarrow \infty$  is arbitrary and since a solution  $\zeta(t)$  of Eq. (6) is weakly unique, the proof of Theorem 2.3 is complete.

**Proof of Theorem 2.4.** It is clear that

$$\beta_T^{(2)}(t) = \int_0^t g_0(\zeta_T(s)) d\eta_T(s) + \gamma_T(t), \quad (18)$$

where

$$\gamma_T(t) = \int_0^t q_T(\xi_T(s)) dW_T(s), \quad q_T(x) = g_T(x) - g_0(G_T(x))G_T'(x).$$

The process  $\gamma_T(t)$  is a continuous martingale with the quadratic characteristics

$$\langle \gamma_T \rangle(t) = \int_0^t q_T^2(\xi_T(s)) ds \quad \text{for all } T > 0.$$

According to the conditions of Theorem 2.4, the functions  $q_T^2(x)$  satisfy the conditions of Lemma 4.2. Thus, for any  $L > 0$ , we have the convergence  $\langle \gamma_T \rangle(L) \xrightarrow{P} 0$  as  $T \rightarrow \infty$ .

The following inequality holds for any constants  $\varepsilon > 0$  and  $\delta > 0$ :

$$P\left\{\sup_{0 \leq t \leq L} |\gamma_T(t)| > \varepsilon\right\} \leq \delta + P\{\langle \gamma_T \rangle(L) > \varepsilon^2 \delta\}$$

(see [2], §3, Theorem 2), which implies the relation

$$\sup_{0 \leq t \leq L} |\gamma_T(t)| \xrightarrow{P} 0 \quad (19)$$

as  $T \rightarrow \infty$ .

Then, similarly to representation (11), on a certain probability space  $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{P})$ , for an arbitrary subsequence  $T_n$ , we get the equality

$$\tilde{\beta}_{T_n}^{(2)}(t) = \int_0^t g_0(\tilde{\zeta}_{T_n}(s)) d\tilde{\eta}_{T_n}(s) + \tilde{\gamma}_{T_n}(t),$$

where

$$\sup_{0 \leq t \leq L} |\tilde{\zeta}_{T_n}(t) - \tilde{\zeta}(t)| \xrightarrow{\tilde{P}} 0, \quad \sup_{0 \leq t \leq L} |\tilde{\eta}_{T_n}(t) - \tilde{\eta}(t)| \xrightarrow{\tilde{P}} 0, \quad \sup_{0 \leq t \leq L} |\tilde{\gamma}_{T_n}(t)| \xrightarrow{\tilde{P}} 0$$

as  $T_n \rightarrow \infty$  for any  $L > 0$ , where the process  $(\tilde{\zeta}(t), \hat{W}(t))$  satisfies Eq. (6),  $\tilde{\eta}(t)$  is defined in (13), and the processes  $\tilde{\beta}_{T_n}^{(2)}(t)$  and  $\beta_{T_n}^{(2)}(t)$  are stochastically equivalent.

Similarly to the proof of convergence (17), we obtain

$$\sup_{0 \leq t \leq L} |\tilde{\beta}_{T_n}^{(2)}(t) - \tilde{\beta}^{(2)}(t)| \xrightarrow{\tilde{P}} 0$$

as  $T_n \rightarrow \infty$ , where

$$\begin{aligned} \tilde{\beta}^{(2)}(t) &= \int_0^t g_0(\tilde{\zeta}(s)) d\tilde{\eta}(s) \\ &= \int_0^t g_0(\tilde{\zeta}(s)) d\tilde{\zeta}(s) - \int_0^t g_0(\tilde{\zeta}(s)) a_0(\tilde{\zeta}(s)) ds. \end{aligned}$$

Thus, the process  $\tilde{\beta}_{T_n}^{(2)}(t)$  weakly converges, as  $T_n \rightarrow \infty$ , to the process  $\tilde{\beta}^{(2)}(t)$ . Since the subsequence  $T_n \rightarrow \infty$  is arbitrary and since the processes  $\tilde{\beta}_{T_n}^{(2)}(t)$  and  $\beta_{T_n}^{(2)}(t)$  are stochastically equivalent, the proof of Theorem 2.4 is complete.

Proof of Theorem 2.5. It is clear that

$$I_T(t) = F_0(\zeta_T(t)) + \int_0^t g_0(\zeta_T(s)) d\eta_T(s) + \alpha_T(t) + \gamma_T(t),$$

where

$$\alpha_T(t) = F_T(\xi_T(t)) - F_0(\zeta_T(t)), \quad \eta_T(t) = \int_0^t G'_T(\xi_T(s)) dW_T(s),$$

$$\gamma_T(t) = \int_0^t q_T(\xi_T(s)) dW_T(s), \quad q_T(x) = g_T(x) - g_0(G_T(x))G'_T(x).$$

Denote, as before,  $P_{NT} = \mathbb{P}\{\sup_{0 \leq t \leq L} |\xi_T(t)| > N\}$ . Since for any constants  $\varepsilon > 0$ ,  $N > 0$ , and  $L > 0$ , we have the inequality

$$\begin{aligned} & \mathbb{P}\left\{\sup_{0 \leq t \leq L} |F_T(\xi_T(t)) - F_0(G_T(\xi_T(t)))| > \varepsilon\right\} \\ & \leq P_{NT} + \frac{2}{\varepsilon} \mathbb{E} \sup_{0 \leq t \leq L} |F_T(\xi_T(t)) - F_0(G_T(\xi_T(t)))| \chi_{\{|\xi_T(t)| \leq N\}} \\ & \leq P_{NT} + \frac{2}{\varepsilon} \sup_{|x| \leq N} |F_T(x) - F_0(G_T(x))|, \end{aligned}$$

we can apply conditions of Theorem 2.5 and convergence (15) to get that

$$\sup_{0 \leq t \leq L} |\alpha_T(t)| \xrightarrow{\mathbb{P}} 0$$

as  $T \rightarrow \infty$ . The proof of the fact that, for  $\gamma_T(t)$ , an analogue of convergence (19) holds is literally the same as in the proof of Theorem 2.4. Then, we can apply Skorokhod's convergent subsequence principle to the process  $(\zeta_T(t), \eta_T(t), \alpha_T(t), \gamma_T(t))$  and, similarly to representation (11), obtain the following equality for an arbitrary subsequence  $T_n$  in a certain probability space  $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}})$ :

$$\tilde{I}_{T_n}(t) = F_0(\tilde{\zeta}_{T_n}(t)) + \int_0^t g_0(\tilde{\zeta}_{T_n}(s)) d\tilde{\eta}_{T_n}(s) + \tilde{\alpha}_{T_n}(t) + \tilde{\gamma}_{T_n}(t),$$

where, as  $T_n \rightarrow \infty$ , for any  $L > 0$ ,

$$\begin{aligned} & \sup_{0 \leq t \leq L} |\tilde{\zeta}_{T_n}(t) - \tilde{\zeta}(t)| \xrightarrow{\tilde{\mathbb{P}}} 0, \quad \sup_{0 \leq t \leq L} |\tilde{\eta}_{T_n}(t) - \tilde{\eta}(t)| \xrightarrow{\tilde{\mathbb{P}}} 0, \\ & \sup_{0 \leq t \leq L} |\tilde{\alpha}_{T_n}(t)| \xrightarrow{\tilde{\mathbb{P}}} 0, \quad \sup_{0 \leq t \leq L} |\tilde{\gamma}_{T_n}(t)| \xrightarrow{\tilde{\mathbb{P}}} 0. \end{aligned}$$

To complete the proof of Theorem 2.5, we repeat the same arguments as in the proof of Theorem 2.4.

**Proof of Theorem 2.6.** According to the Itô formula, the process  $\zeta_T(t) = f_T(\xi_T(t))$  satisfies the equation  $d\zeta_T(t) = \hat{\sigma}_T(\zeta_T(t)) dW_T(t)$ , where  $\hat{\sigma}_T(x) = f'_T(\varphi_T(x))$ ,  $\varphi_T(x)$  is the inverse function of the function  $f_T(x)$ , and  $\zeta_T(0) = f_T(x_0) \rightarrow y_0$  as  $T \rightarrow \infty$ . In addition, the following equality holds:

$$I_T(t) = \hat{F}_T(\zeta_T(t)) + \int_0^t \hat{g}_T(\zeta_T(s)) d\zeta_T(s),$$

where  $\hat{F}_T(x) = F_T(\varphi_T(x))$  and  $\hat{g}_T(x) = g_T(\varphi_T(x)) \cdot \hat{\sigma}_T^{-1}(x)$ .

It is easy to see that condition 1 of the present theorem implies that

$$\int_0^x \frac{dv}{\hat{\sigma}_T^2(v)} \rightarrow \int_0^x \frac{dv}{\sigma_0^2(v)}$$

as  $T \rightarrow \infty$  for all  $x$ , whereas condition 2 implies that

$$\sup_{|x| \leq N} |\hat{F}_T(x) + c_T^{(2)} + c_T^{(1)}x - F_0(x)| \rightarrow 0$$

and

$$\int_{-N}^N |\hat{g}_T(x) - c_T^{(1)} - g_0(x)|^2 dx \rightarrow 0$$

as  $T \rightarrow \infty$  for any  $N > 0$ .

This means that the necessary and sufficient conditions of weak convergence of the process  $(\zeta_T(t), I_T(t))$  as  $T \rightarrow \infty$  to the process  $(\zeta(t), I(t))$  from [11] hold with  $b_T = c_T^{(1)}$  and  $a_T = c_T^{(2)}$ .

#### 4 Auxiliary results

**Lemma 4.1.** *Let  $\xi_T$  be a solution of Eq. (1) from the class  $K(G_T)$ . Then, for any  $N > 0$  and any Borel set  $B \subset [-N; N]$ , there exists a constant  $C_L$  such that*

$$\int_0^L \mathbb{P}\{G_T(\xi_T(s)) \in B\} ds \leq C_L \psi(\lambda(B)),$$

where  $\lambda(B)$  is the Lebesgue measure of  $B$ , and  $\psi(|x|)$  is a bounded function satisfying  $\psi(|x|) \rightarrow 0$  as  $|x| \rightarrow 0$ .

**Proof.** Consider the function

$$\Phi_T(x) = 2 \int_0^x f'_T(u) \left( \int_0^u \frac{\chi_B(G_T(v))}{f'_T(v)} dv \right) du.$$

The function  $\Phi_T(x)$  is continuous, the derivative  $\Phi'_T(x)$  of this function is continuous, and the second derivative  $\Phi''_T(x)$  exists a.e. with respect to the Lebesgue measure and is locally bounded. Therefore, we can apply the Itô formula to the process  $\Phi_T(\xi_T(t))$ , where  $\xi_T(t)$  is a solution of Eq. (1).

Furthermore,

$$\Phi'_T(x)a_T(x) + \frac{1}{2}\Phi''_T(x) = \chi_B(x)$$

a.e. with respect to the Lebesgue measure. Using the latter equality, we conclude that

$$\int_0^t \chi_B(\zeta_T(s)) ds = \Phi_T(\xi_T(t)) - \Phi_T(x_0) - \int_0^t \Phi'_T(\xi_T(s)) dW_T(s) \quad (20)$$

with probability one for all  $t \geq 0$ , where  $\zeta_T(t) = G_T(\xi_T(t))$ . Hence, using the properties of stochastic integrals, we obtain that

$$\int_0^t \mathbb{P}\{\zeta_T(s) \in B\} ds = \mathbb{E}[\Phi_T(\xi_T(t)) - \Phi_T(x_0)]. \quad (21)$$

According to condition  $(A_2)$  and inequality  $|G_T(x)| \geq C|x|^\alpha$ ,  $C > 0$ ,  $\alpha > 0$ , we have

$$|\Phi_T(x) - \Phi_T(x_0)| \leq 2\psi(\lambda(B)) [1 + |x|^m] \leq 2\psi(\lambda(B)) [1 + C^{-\frac{m}{\alpha}} |G_T(x)|^{\frac{m}{\alpha}}].$$

Hence, using inequality (5), we obtain that

$$\left| \mathbb{E}[\Phi_T(\xi_T(L)) - \Phi_T(x_0)] \right| \leq C_L \psi(\lambda(B))$$

for some constant  $C_L$ . The latter inequality and Eq. (21) prove Lemma 4.1.  $\square$

**Lemma 4.2.** *Let  $\xi_T$  be a solution of Eq. (1) from the class  $K(G_T)$ . If, for measurable locally bounded functions  $q_T(x)$ , condition  $(A_3)$  holds, then, for any  $L > 0$ ,*

$$\sup_{0 \leq t \leq L} \left| \int_0^t q_T(\xi_T(s)) ds \right| \xrightarrow{\mathbb{P}} 0$$

as  $T \rightarrow \infty$ .

**Proof.** Consider the function

$$\Phi_T(x) = 2 \int_0^x f'_T(u) \left( \int_0^u \frac{q_T(v)}{f'_T(v)} dv \right) du.$$

The same arguments as used to obtain Eq. (20) yield that

$$\int_0^t q_T(\xi_T(s)) ds = \Phi_T(\xi_T(t)) - \Phi_T(x_0) - \int_0^t \Phi'_T(\xi_T(s)) dW_T(s). \quad (22)$$

It is clear that, for any constants  $\varepsilon > 0$ ,  $N > 0$ , and  $L > 0$ , we have the inequalities

$$\begin{aligned} \mathbb{P} \left\{ \sup_{0 \leq t \leq L} |\Phi_T(\xi_T(t))| > \varepsilon \right\} &\leq P_{NT} + \frac{2}{\varepsilon} \int_{-N}^N f'_T(u) \left| \int_0^u \frac{q_T(v)}{f'_T(v)} dv \right| du, \\ \mathbb{P} \left\{ \sup_{0 \leq t \leq L} \left| \int_0^t \Phi'_T(\xi_T(s)) dW_T(s) \right| > \varepsilon \right\} \\ &\leq P_{NT} + \frac{1}{\varepsilon^2} \mathbb{E} \sup_{0 \leq t \leq L} \left| \int_0^t \Phi'_T(\xi_T(s)) \chi_{\{|\xi_T(s)| \leq N\}} dW_T(s) \right|^2 \\ &\leq P_{NT} + \frac{4}{\varepsilon^2} \mathbb{E} \int_0^L [\Phi'_T(\xi_T(s))]^2 \chi_{\{|\xi_T(s)| \leq N\}} ds \\ &\leq P_{NT} + \frac{16}{\varepsilon^2} L \sup_{|x| \leq N} \left[ f'_T(x) \left| \int_0^x \frac{q_T(v)}{f'_T(v)} dv \right| \right]^2, \end{aligned}$$

where  $P_{NT} = \mathbb{P}\{\sup_{0 \leq t \leq L} |\xi_T(t)| > N\}$ .

Therefore, using convergence (15), we obtain

$$\sup_{0 \leq t \leq L} |\Phi_T(\xi_T(t)) - \Phi_T(x_0)| \xrightarrow{\mathbb{P}} 0$$

and

$$\sup_{0 \leq t \leq L} \left| \int_0^t \Phi'_T(\xi_T(s)) dW_T(s) \right| \xrightarrow{\mathbb{P}} 0$$

as  $T \rightarrow \infty$ . Thus, Eq. (22) implies the statement of Lemma 4.2.  $\square$

**Lemma 4.3.** *Let  $\xi_T$  be a solution of Eq. (1) from the class  $K(G_T)$ , and let  $\zeta_T(t) = G_T(\xi_T(t)) \xrightarrow{\mathbb{P}} \zeta(t)$  as  $T \rightarrow \infty$ . Then for any measurable locally bounded function  $g(x)$ , we have the convergence*

$$\sup_{0 \leq t \leq L} \left| \int_0^t g(\zeta_T(s)) ds - \int_0^t g(\zeta(s)) ds \right| \xrightarrow{\mathbb{P}} 0$$

as  $T \rightarrow \infty$  for any constant  $L > 0$ .

**Proof.** Let  $\varphi_N(x) = 1$  for  $|x| \leq N$ ,  $\varphi_N(x) = N + 1 - |x|$  for  $|x| \in [N, N + 1]$ , and  $\varphi_N(x) = 0$  for  $|x| > N + 1$ . Then, for all  $T > 0$  and  $L > 0$ ,

$$\begin{aligned} & \mathbb{P} \left\{ \sup_{0 \leq t \leq L} \left| \int_0^t [g(\zeta_T(s)) - g(\zeta_T(s))\varphi_N(\zeta_T(s))] ds \right| > 0 \right\} \leq \mathbb{P} \left\{ \sup_{0 \leq t \leq L} |\zeta_T(t)| > N \right\}, \\ & \mathbb{P} \left\{ \sup_{0 \leq t \leq L} \left| \int_0^t [g(\zeta(s)) - g(\zeta(s))\varphi_N(\zeta(s))] ds \right| > 0 \right\} \\ & \leq \mathbb{P} \left\{ \sup_{0 \leq t \leq L} |\zeta(t)| > N \right\} \leq \overline{\lim}_{T \rightarrow \infty} \mathbb{P} \left\{ \sup_{0 \leq t \leq L} |\zeta_T(t)| > N \right\}. \end{aligned}$$

According to Theorem 2.1, convergence (4) holds for the process  $\zeta_T(t)$ . So, to complete the proof of Lemma 4.3, we need to establish that

$$\int_0^L |g(\zeta_T(s))\varphi_N(\zeta_T(s)) - g(\zeta(s))\varphi_N(\zeta(s))| ds \xrightarrow{\mathbb{P}} 0 \quad (23)$$

as  $T \rightarrow \infty$ .

First, assume that the function  $g(x)$  is continuous. Then

$$g(\zeta_T(s))\varphi_N(\zeta_T(s)) - g(\zeta(s))\varphi_N(\zeta(s)) \xrightarrow{\mathbb{P}} 0$$

as  $T \rightarrow \infty$  for all  $0 \leq s \leq L$ , and  $|g(x)\varphi_N(x)| \leq C_N$  for all  $x$ . Thus, by Lebesgue's dominated convergence theorem we have convergence (23). Second, let the function  $g(x)$  be measurable and locally bounded. Then, using Luzin's theorem, we conclude that, for any  $\delta > 0$ , there exists a continuous function  $g^\delta(x)$  that coincides with  $g(x)$  for  $x \notin B^\delta$ , where  $B^\delta \subset [-N - 1, N + 1]$ , and the Lebesgue measure satisfies the inequality  $\lambda(B^\delta) < \delta$ . Thus, for every  $\delta > 0$ , convergence (23) holds for the function  $g^\delta(x)$ . Since, for any  $\varepsilon > 0$ ,

$$\begin{aligned} & \mathbb{P} \left\{ \int_0^L |g(\zeta_T(s))\varphi_N(\zeta_T(s)) - g^\delta(\zeta_T(s))\varphi_N(\zeta_T(s))| ds > \varepsilon \right\} \\ & \leq \frac{1}{\varepsilon} \mathbb{E} \int_0^L |g(\zeta_T(s))\varphi_N(\zeta_T(s)) - g^\delta(\zeta_T(s))\varphi_N(\zeta_T(s))| \chi_{\{B^\delta\}}(\zeta_T(s)) ds \\ & \leq \frac{C_N}{\varepsilon} \int_0^L \mathbb{P} \{ \zeta_T(s) \in B^\delta \} ds, \\ & \mathbb{P} \left\{ \int_0^L |g(\zeta(s))\varphi_N(\zeta(s)) - g^\delta(\zeta(s))\varphi_N(\zeta(s))| ds > \varepsilon \right\} \\ & \leq \frac{C_N}{\varepsilon} \int_0^L \mathbb{P} \{ \zeta(s) \in B^\delta \} ds \leq \frac{C_N}{\varepsilon} \overline{\lim}_{T \rightarrow \infty} \int_0^L \mathbb{P} \{ \zeta_T(s) \in B^\delta \} ds, \end{aligned}$$

taking into account Lemma 4.1, we conclude that convergence (23) holds for such a function  $g(x)$  as well.  $\square$



**Lemma 4.4.** Let  $\xi_T$  be a solution of Eq. (1) from the class  $K(G_T)$ , and let  $\zeta_T(t) = G_T(\xi_T(t)) \xrightarrow{P} \zeta(t)$  and  $\eta_T(t) = \int_0^t G'_T(\xi_T(s)) dW_T(s) \xrightarrow{P} \eta(t)$  as  $T \rightarrow \infty$ . Then, for measurable locally bounded functions  $g(x)$ , we have the convergence

$$\sup_{0 \leq t \leq L} \left| \int_0^t g(\zeta_T(s)) d\eta_T(s) - \int_0^t g(\zeta(s)) d\eta(s) \right| \xrightarrow{P} 0$$

as  $T \rightarrow \infty$  for any constant  $L > 0$ .

**Proof.** Similarly to the proof of Lemma 4.3, it suffices to obtain an analogue of convergence (23), that is, to get that, for any  $N > 0$  and  $L > 0$ ,

$$\sup_{0 \leq t \leq L} \left| \int_0^t g(\zeta_T(s)) \varphi_N(\zeta_T(s)) d\eta_T(s) - \int_0^t g(\zeta(s)) \varphi_N(\zeta(s)) d\eta(s) \right| \xrightarrow{P} 0 \quad (24)$$

as  $T \rightarrow \infty$ , where  $\varphi_N(x)$  is defined in the proof of Lemma 4.3. The proof of convergence (24) for a continuous function  $g(x)$  is similar to that of the corresponding theorem in [17], Chapter 2, §6. The explicit form of the quadratic characteristic  $\langle \eta_T \rangle(t)$  of the martingale  $\eta_T(t)$  and condition  $(A_1)$  imply the inequality

$$\int_0^L [\varphi_N(\zeta_T(t))]^2 d\langle \eta_T \rangle(t) \leq C_N L,$$

which is used for the proof of convergence (24). The extension of such a convergence to the class of measurable locally bounded functions is based on Lemma 4.1 and is provided similarly to the proof of Lemma 4.3.  $\square$

**Lemma 4.5.** Let  $\xi_T$  be a solution of Eq. (1) belonging to the class  $K(G_T)$ , and let the stochastic process  $(\zeta_T(t), \eta_T(t))$ , with  $\zeta_T(t) = G_T(\xi_T(t))$  and  $\eta_T(t) = \int_0^t G'_T(\xi_T(s)) dW_T(s)$  be stochastically equivalent to the process  $(\tilde{\zeta}_T(t), \tilde{\eta}_T(t))$ . Then the process

$$\int_0^t g(\zeta_T(s)) ds + \int_0^t q(\zeta_T(s)) d\eta_T(s),$$

where  $g(x)$  and  $q(x)$  are measurable locally bounded functions, is stochastically equivalent to the process

$$\int_0^t g(\tilde{\zeta}_T(s)) ds + \int_0^t q(\tilde{\zeta}_T(s)) d\tilde{\eta}_T(s).$$

**Proof.** The proof is the same as that of Theorem 2 from [3].  $\square$

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